2
Number Systems

Objectives
After studying this chapter, the student should be able to:

- Understand the concept of number systems.
- Distinguish between non-positional and positional number systems.
- Describe the decimal, binary, hexadecimal and octal system.
- Convert a number in binary, octal or hexadecimal to a number in the decimal system.
- Convert a number in the decimal system to a number in binary, octal and hexadecimal.
- Convert a number in binary to octal and vice versa.
- Convert a number in binary to hexadecimal and vice versa.
- Find the number of digits needed in each system to represent a particular value.
2-1 Introduction

A number system defines how a number can be represented using distinct symbols. A number can be represented differently in different systems. For example, the two numbers (2A)\(_16\) and (52)\(_8\) both refer to the same quantity, (42)\(_{10}\), but their representations are different.

\[(2A)_{16} = (52)_{8} = (42)_{10}\]

Several number systems have been used in the past and can be categorized into two groups: positional and non-positional systems. Our main goal is to discuss the positional number systems, but we also give examples of non-positional systems.

2-2 Positional Number Systems

In a positional number system, the position a symbol occupies in the number determines the value it represents. In this system, a number represented as:

\[\pm (S_{k-1} \ldots S_2 S_1 S_0. S_{-1} S_{-2} \ldots S_{-f})_b\]

has the value of:

\[n = \pm \sum_{i=-f}^{k-1} S_i \times b^i\]

in which \(S\) is the set of symbols, \(b\) is the base (or radix).

Example: \((42)_{10}\)
The Decimal System (base 10)

The word decimal is derived from the Latin root *decem* (ten). In this system the base $b = 10$ and we use ten symbols

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

The symbols in this system are often referred to as *decimal digits* or just *digits*.

Different Systems

**Decimal:** $0, \ldots, 7, 8, 9, 10, 11, \ldots$

**Octal:** $0, \ldots, 7, 10, 11, \ldots, 16, 17, 20, 21, \ldots$

**Binary:** $0, 1, 10, 11, 100, 101, 110, 111, 1000, \ldots$

Integers (base 10)

<table>
<thead>
<tr>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$\ldots$</th>
<th>$10^2$</th>
<th>$10^3$</th>
<th>$10^4$</th>
<th>Place values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm S_{n-1}$</td>
<td>$S_{n-2}$</td>
<td>$\ldots$</td>
<td>$S_2$</td>
<td>$S_1$</td>
<td>$S_0$</td>
<td>Number</td>
</tr>
</tbody>
</table>

$$N = \pm S_{n-1} \times 10^{n-1} + S_{n-2} \times 10^{n-2} + \ldots + S_2 \times 10^2 + S_1 \times 10^1 + S_0 \times 10^0$$

Values

Example 2.1

The following shows the place values for the integer $+224$ in the decimal system.

$$N = 2 \times 10^2 + 2 \times 10^1 + 4 \times 10^0$$

Place values

Number

Values
Example 2.2

The following shows the place values for the decimal number $-7508$. We have used 1, 10, 100, and 1000 instead of powers of 10.

\[
N = -\left(7 \times 1000 + 5 \times 100 + 0 \times 10 + 8 \times 1\right)
\]

Reals (base 10)

\[
R = S_{k-1} \times 10^{k-1} + \ldots + S_1 \times 10^1 + S_0 \times 10^0 + S_{-1} \times 10^{-1} + \ldots + S_{-l} \times 10^{-l}
\]

Example 2.3

The following shows the place values for the real number $+24.13$. 

\[
R = +\left(2 \times 10 + 4 \times 1 + 1 \times 0.1 + 3 \times 0.01\right)
\]
The Binary System (base 2)

The word binary is derived from the Latin root *bini* (or two by two). In this system the base $b = 2$ and we use only two symbols,

$$S = \{0, 1\}$$

The symbols in this system are often referred to as binary digits or bits (binary digit).

Integers (base 2)

Example 2.4

The following shows that the number $(11001)_2$ in binary is the same as 25 in decimal. The subscript 2 shows that the base is 2.

$$N = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

The equivalent decimal number is $N = 16 + 8 + 0 + 0 + 1 = 25$. 

2.10
Reals (base 2)

\[ R = \pm \sum_{k=1}^{\infty} S_k \times 2^{k-1} + \ldots + S_1 \times 2^1 + S_0 \times 2^0 + \sum_{l=-1}^{-\infty} S_{l-1} \times 2^{-l} \]

Example 2.5

The following shows that the number \((101.11)_2\) in binary is equal to the number 5.75 in decimal.

\[
\begin{array}{cccccc}
2^2 & 2^1 & 2^0 & 2^{-1} & 2^{-2} & \text{Place values} \\
1 & 0 & 1 & \cdot & 1 & \text{Number} \\
\text{R} & = & 1 \times 2^2 & + & 0 \times 2^1 & + 1 \times 2^0 & + 1 \times 2^{-1} & + 1 \times 2^{-2} & \text{Values} \\
\end{array}
\]

The decimal number is \( N = 4 + 0 + 1 + 0.5 + 0.25 = 5.75 \)

The Hexadecimal System (base 16)

The word hexadecimal is derived from the Greek root hex (six) and the Latin root decem (ten). In this system the base \( b = 16 \) and we use sixteen symbols to represent a number. The set of symbols is

\[ S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\} \]

Note that the symbols A, B, C, D, E, F are equivalent to 10, 11, 12, 13, 14, and 15 respectively. The symbols in this system are often referred to as hexadecimal digits.
Integers (base 16)

Example 2.6

The following shows that the number $(2AE)_{16}$ in hexadecimal is equivalent to 686 in decimal.

The Octal System (base 8)

The word octal is derived from the Latin root octo (eight). In this system the base $b = 8$ and we use eight symbols to represent a number. The set of symbols is

$$S = \{0, 1, 2, 3, 4, 5, 6, 7\}$$
Integers (base 8)

\[
\begin{array}{cccccc}
8^3 & 8^2 & \cdots & 8^1 & 8^0 \\
S_3 & S_2 & \cdots & S_1 & S_0 \\
\downarrow & \downarrow & \cdots & \downarrow & \downarrow \\
\pm S_{x,1} \times 8^{x-1} + S_{x,2} \times 8^{x-2} & + \cdots & + S_1 \times 8^1 & + S_0 \times 8^0 \\
N = & \end{array}
\]

Example 2.7

The following shows that the number \((1256)_8\) in octal is the same as 686 in decimal.

\[
\begin{align*}
N &= 1 \times 8^3 + 2 \times 8^2 + 5 \times 8^1 + 6 \times 8^0 \\
&= 512 + 128 + 40 + 6 = 686
\end{align*}
\]

Summary of Four Positional Systems

Table 2.1 shows a summary of the four positional number systems discussed in this chapter.

<table>
<thead>
<tr>
<th>System</th>
<th>Base</th>
<th>Symbols</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal</td>
<td>10</td>
<td>0, 1, 2, 3, 4, 5, 6, 7, 8, 9</td>
<td>2345.56</td>
</tr>
<tr>
<td>Binary</td>
<td>2</td>
<td>0, 1</td>
<td>(1001.11)_2</td>
</tr>
<tr>
<td>Octal</td>
<td>8</td>
<td>0, 1, 2, 3, 4, 5, 6, 7</td>
<td>(156.23)_8</td>
</tr>
<tr>
<td>Hexadecimal</td>
<td>16</td>
<td>0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F</td>
<td>(A2C.A1)_{16}</td>
</tr>
</tbody>
</table>
Table 2.2 shows how the number 0 to 15 is represented in different systems.

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Binary</th>
<th>Octal</th>
<th>Hexadecimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>12</td>
<td>A</td>
</tr>
<tr>
<td>11</td>
<td>1011</td>
<td>13</td>
<td>B</td>
</tr>
<tr>
<td>12</td>
<td>1100</td>
<td>14</td>
<td>C</td>
</tr>
<tr>
<td>13</td>
<td>1101</td>
<td>15</td>
<td>D</td>
</tr>
<tr>
<td>14</td>
<td>1110</td>
<td>16</td>
<td>E</td>
</tr>
<tr>
<td>15</td>
<td>1111</td>
<td>17</td>
<td>F</td>
</tr>
</tbody>
</table>

**Conversion**

We need to know how to convert a number in one system to the equivalent number in another system. Since the decimal system is more familiar than the other systems, we first show how to covert from any base to decimal. Then we show how to convert from decimal to any base. Finally, we show how we can easily convert from binary to hexadecimal or octal and vice versa.
**Any Base to Decimal Conversion**

![Diagram of Any Base to Decimal Conversion](image)

**Figure 2.5** Converting other bases to decimal

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**Example 2.8**

The following shows how to convert the binary number \((110.11)_2\) to decimal: \((110.11)_2 = 6.75\).

<table>
<thead>
<tr>
<th>Binary</th>
<th>Place values</th>
<th>Partial results</th>
<th>Decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2(^2)</td>
<td>4</td>
<td>6.75</td>
</tr>
<tr>
<td>1</td>
<td>2(^1)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2(^0)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>2(^{-1})</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>2(^{-2})</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

2.20
Example 2.9

The following shows how to convert the hexadecimal number (1A.23)\(_{16}\) to decimal.

<table>
<thead>
<tr>
<th>Hexadecimal Place values</th>
<th>Partial result</th>
<th>Decimal: 26.137</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (16^1)</td>
<td>16 + 10</td>
<td></td>
</tr>
<tr>
<td>A (16^0)</td>
<td>0.125</td>
<td></td>
</tr>
<tr>
<td>(0.23) (16^{-1})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3) (16^{-2})</td>
<td>0.012</td>
<td></td>
</tr>
</tbody>
</table>

Note that the result in the decimal notation is not exact, because \(3 \times 16^{-2} = 0.01171875\). We have rounded this value to three digits (0.012).

Example 2.10

The following shows how to convert (23.17)\(_{8}\) to decimal.

<table>
<thead>
<tr>
<th>Octal Place values</th>
<th>Partial result</th>
<th>Decimal: 19.234</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (8^1)</td>
<td>16 + 3</td>
<td></td>
</tr>
<tr>
<td>3 (8^0)</td>
<td>0.125</td>
<td></td>
</tr>
<tr>
<td>(0.17) (8^{-1})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7) (8^{-2})</td>
<td>0.109</td>
<td></td>
</tr>
</tbody>
</table>

This means that (23.17)\(_{8}\) \(\approx 19.234\) in decimal. Again, we have rounded up \(7 \times 8^{-2} = 0.109375\).
Decimal to Any Base (Integral part)

**Example 2.11**

Convert 35 in decimal to binary, we start with the number in decimal and move to the left while continuously finding the quotients and the remainder of division by 2. The result is \(35 = (100011)_2\).

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>100011</td>
</tr>
</tbody>
</table>

**Example 2.12**

The following shows how to convert 126 in decimal to its equivalent in the octal system. We move to the right while continuously finding the quotients and the remainder of division by 8. The result is \(126 = (176)_8\).

<table>
<thead>
<tr>
<th>Decimal</th>
<th>Octal</th>
</tr>
</thead>
<tbody>
<tr>
<td>126</td>
<td>176</td>
</tr>
</tbody>
</table>


 Q: Quotients  
 R: Remainders  
 S: Source  
 D: Destination  
 \(D_j\): Destination digit
Example 2.13

The following shows how we convert 126 in decimal to its equivalent in the hexadecimal system. We move to the right while continuously finding the quotients and the remainder of division by 16. The result is $126 = (7E)_{16}$

Example 2.14

Convert the decimal number 0.625 to binary. $0.625 = (0.101)_{2}$
Example 2.15
The following shows how to convert 0.634 to octal using a maximum of four digits. The result is $0.634 = (0.5044)_8$. Note that we multiple by 8 (base octal).

<table>
<thead>
<tr>
<th>Decimal</th>
<th>0.634</th>
<th>0.072</th>
<th>0.576</th>
<th>0.608</th>
<th>0.864</th>
</tr>
</thead>
<tbody>
<tr>
<td>Octal</td>
<td>5</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Example 2.16
The following shows how to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point. The result is $178.6 = (B2.9)_{16}$. Note that we divide or multiple by 16 (base hexadecimal).

<table>
<thead>
<tr>
<th>Decimal</th>
<th>0</th>
<th>11</th>
<th>178</th>
<th>0.6</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hexadecimal</td>
<td>B</td>
<td>2</td>
<td>9</td>
<td>•</td>
<td>•</td>
</tr>
</tbody>
</table>
Example 2.17

An alternative method for converting a small decimal integer (usually less than 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shown:

<table>
<thead>
<tr>
<th>Place values</th>
<th>$2^7$</th>
<th>$2^6$</th>
<th>$2^5$</th>
<th>$2^4$</th>
<th>$2^3$</th>
<th>$2^2$</th>
<th>$2^1$</th>
<th>$2^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Decimal 165 = 128 + 0 + 32 + 0 + 0 + 4 + 0 + 1

Binary | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |

Example 2.18

A similar method can be used to convert a decimal fraction to binary when the denominator is a power of two:

<table>
<thead>
<tr>
<th>Place values</th>
<th>$2^{-1}$</th>
<th>$2^{-2}$</th>
<th>$2^{-3}$</th>
<th>$2^{-4}$</th>
<th>$2^{-5}$</th>
<th>$2^{-6}$</th>
<th>$2^{-7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal</td>
<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>1/16</td>
<td>1/32</td>
<td>1/64</td>
<td>1/128</td>
</tr>
</tbody>
</table>

Decimal = 27/64 = 16/64 + 8/64 + 2/64 + 1/64

Decimal 27/64 = 0 + 1/4 + 1/8 + 0 + 1/32 + 1/64

Binary | 0 | 1 | 1 | 0 | 1 | 1 |

The answer is then $(0.011011)_2$
**Binary-Hexadecimal Conversion**

<table>
<thead>
<tr>
<th>$B_i$: Binary digit (bit)</th>
<th>$H_j$: Hexadecimal digit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_m$, $B_{m-1}$, $B_{m-2}$, $B_{m-3}$</td>
<td>$H_n$, $H_{j-1}$, $H_{j-2}$, $H_{j-3}$</td>
</tr>
</tbody>
</table>

Example 2.19

Show the hexadecimal equivalent of the binary number $(10011100010)_2$.

<table>
<thead>
<tr>
<th>100</th>
<th>1110</th>
<th>0010</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>E</td>
<td>216</td>
</tr>
</tbody>
</table>

Example 2.20

What is the binary equivalent of $(24C)_{16}$?

**Solution**

Each hexadecimal digit is converted to 4-bit patterns:

- $2 \rightarrow 0010$
- $4 \rightarrow 0100$
- $C \rightarrow 1100$

The result is $(001001001100)_2$. 

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2.31

2.32
Binary-Octal Conversion

$B_i$: Binary digit (bit)  $O_j$: Octal digit

\[
\begin{array}{ccc}
B_m & B_{m-1} & B_{m-2} \\
\downarrow & & \downarrow \\
O_n & \cdots & O_1
\end{array}
\]

Example 2.21

Show the octal equivalent of the binary number $(101110010)_2$.

\[
\begin{array}{ccc}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
5 & 6 & 2 \\
8 & & \\
\end{array}
\]

Example 2.22

What is the binary equivalent of $(24)_8$?

Solution

Write each octal digit as its equivalent bit pattern to get

\[
2 \rightarrow 010 \quad \text{and} \quad 4 \rightarrow 100
\]

The result is $(0101100)_2$. 
Octal-Hexadecimal Conversion

![Diagram of Octal to Hexadecimal and Hexadecimal to Octal Conversion]

**Figure 2.12** Octal to hexadecimal and hexadecimal to octal conversion

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**Example 2.23**

Find the minimum number of binary digits required to store decimal integers with a maximum of six digits.

**Solution**

\[ k = 6, \ b_1 = 10, \text{ and } b_2 = 2. \text{ Then} \]

\[ x = \left\lceil k \times \frac{\log b_1}{\log b_2} \right\rceil = \left\lceil 6 \times \frac{1}{0.30103} \right\rceil = 20. \]

The largest six-digit decimal number is 999,999 (i.e., \(10^6 - 1\)) and the largest 20-bit binary number is 1,048,575 (i.e., \(2^{20} - 1\)). Note that the largest number that can be represented by a 19-bit number is 524,287, which is smaller than 999,999. We definitely need twenty bits.